AD 658672

TECHNICAL REPORT
68-14-OSD

ON INEXTENSIONAL VIBRATIONS OF THIN SHELLS

By
Edward W. Ross, Jr.

July 1967

UNITED STATES ARMY
NATICK LABORATORIES
Natick, Massachusetts 01760

Office of the
Scientific Director

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va. 22151

This document is approved for release and sale; its distribution is unlimited.

AD ____________________

TECHNICAL REPORT
68-14-OSD

ON INEXTENSIONAL VIBRATIONS OF THIN SHELLS

by

Edward W. Ross, Jr.
Mathematician

July 1967

U. S. ARMY NATICK LABORATORIES
Natick, Massachusetts 01760

This document has been approved for public release and sale;its distribution is unlimited.

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

Citation of trade names in this report does not constitute an official endorsement or approval of the use of such items.

Destroy this report when no longer needed. Do not return it to the originator.

CONTENTS

ABSTRACT

In this paper the non-symmetric, free, elastic vibrations of thin domes of revolution are studied. It is assumed that the frequency is low. The asymptotic approximations previously given by the writer are used to estimate the general solution to the shell vibration equations at low frequencies. Approximations for the low natural frequencies and modes are derived systematically under a variety of edge conditions. Low natural frequencies are found only when the edge conditions impose no forces tangent to the shell surface. When the edge is free (and only then) Rayleigh's inextensional frequencies are recovered. For certain other edge conditions new natural frequencies are found that are above Rayleigh's frequencies but still low compared e.g. with the lowest membrane frequency. The displacement modes associated with these new frequencies are mostly of inextensional type. The general results are applied to estimate these new frequencies for spherical domes.

1. INTRODUCTION

The inextensional vibrations of thin shells were first studied by Rayleigh [1][1], and since that time his procedure has often been used to estimate natural frequencies for various shell shapes. The frequencies obtained by this procedure are much lower (for a thin shell) than those predicted by any other method and are therefore of great practical interest in such applications as tents, parachutes, and metal or plastic containers.

However, there are good reasons for skepticism concerning the generality of Rayleigh's procedure. For example, Love [2] has shown that the modes satisfy neither the motion equations nor (with a few exceptions) the edge conditions. Also, Arnold and Warburton [3] observed that Rayleigh's procedure gave good agreement with experiments in some cases, but that changes in the edge conditions could cause enormous changes in the lowest measured frequencies and completely destroy the agreement. It appears, therefore, that we do not understand these vibrations as well as we ought to.

In the present paper we shall show how inextensional modes may be systematically derived from a general theory of shell vibration, without explicitly assuming that the mode is inextensional. Rather, it is merely assumed that the frequency is low (in a sense that will later be made more precise), and from this assumption inextensional modes and frequencies are derived. This change of procedure is important for two

[1]Numbers in brackets designate References at end of paper.

reasons. First, we find that inextensional modes can be derived only for certain edge conditions, and this sheds light on the questions raised in [2] and [3]. Second, the new procedure leaves the way open to find all low frequencies, whereas Rayleigh's procedure is limited to frequencies for which the modal bending energy greatly exceeds the modal stretching energy. For certain edge conditions, we shall find inextensional modes with frequencies different from those obtained by Rayleigh.

To demonstrate the procedure in a context general enough to be convincing but simple enough to avoid unessential manipulations, we consider a general dome of revolution executing small, non-symmetric vibrations. We shall use the approximations obtained by the author [4] to write down an approximate general solution of the differential equation system when the frequency is low. This solution is substituted into the boundary conditions, the resulting frequency determinant is solved and the ratios of the arbitrary constants are found. This entire process is carried through for four different edge conditions, starting with a free edge and proceeding at each stage to the "freest" of the remaining edge conditions. The frequency increases with each new edge condition until we exhaust all edge conditions for which low frequencies can be found.

For the two "freest" edge conditions this procedure gives complete estimates of the mode but only an order-of-magnitude estimate of the frequency. To find frequency estimates we use Rayleigh's Principle for these cases. In the other two cases explicit estimates are found

for the inextensional frequencies. The general formulas are applied to a spherical dome, and numerical results are obtained for the previously unknown inextensional frequencies.

2. FUNDAMENTAL EQUATIONS AND SOLUTIONS

We shall adopt as our starting point the equations of thin-shell theory propounded by Sanders [5] and modified by the inclusion of translational (but not rotational) inertia. All effects of transverse shear and thickness change are omitted in this theory. We may write the system in dimensionless form as in [4].

$$\gamma_{ss} = u' + wr_s^{-1}, \qquad \gamma_{\theta\theta} = uf + vM + wr_\theta^{-1}$$
$$\gamma_{s\theta} = 1/2(v' - uM - vf) \tag{1}$$

$$b_s = -w' + ur_s^{-1}, \qquad b_\theta = vr_\theta^{-1} + wM \tag{2}$$

$$k_{ss} = b_s', \qquad k_{\theta\theta} = fb_s + Mb_\theta$$
$$k_{s\theta} = 1/2\ \{b_\theta' - Mb_s - fb_\theta + 1/2(r_\theta^{-1} - r_s^{-1})(v' + vf + uM)\} \tag{3}$$

$$n_{ss} = \gamma_{ss} + \nu\gamma_{\theta\theta}, \qquad n_{\theta\theta} = \gamma_{\theta\theta} + \nu\gamma_{ss}$$
$$n_{s\theta} = (1 - \nu)\gamma_{s\theta} \tag{4}$$

$$m_{ss} = k_{ss} + \nu k_{\theta\theta}, \qquad m_{\theta\theta} = k_{\theta\theta} + \nu k_{ss}$$
$$m_{s\theta} = (1 - \nu)k_{s\theta} \tag{5}$$

$$q_s = m_{ss}' + f(m_{ss} - m_{\theta\theta}) + Mm_{s\theta}$$
$$q_\theta = m_{s\theta}' + 2fm_{s\theta} - Mm_{\theta\theta} \tag{6}$$

The motion equations are

$$(1 - \nu^2)^{-1}\{n_{ss}{}' + f(n_{ss} - n_{\theta\theta}) + Mn_{s\theta}\} + \Omega^2 u$$
$$+ \epsilon^2\{q_s r_s{}^{-1} + 1/2Mn_{s\theta}(r_s{}^{-1} - r_\theta{}^{-1})\} = 0 \tag{7}$$

$$(1 - \nu^2)^{-1}\{n_{s\theta}{}' + 2fn_{s\theta} - Mn_{\theta\theta}\} + \Omega^2 v$$
$$+ \epsilon^2\{q_\theta r_\theta{}^{-1} + 1/2[(r_\theta{}^{-1} - r_s{}^{-1})M_{s\theta}]'\} = 0 \tag{8}$$

$$(1 - \nu^2)^{-1}\{n_{ss}r_s{}^{-1} + n_{\theta\theta}r_\theta{}^{-1}\} - \Omega^2 w - \epsilon^2\{q_s{}' + fq_s + Mq_\theta\} = 0 \tag{9}$$

Here in dimensionless form u, v, and w are the meridional, circumferential, and normal displacements, γ_{ss}, $\gamma_{\theta\theta}$ and $\gamma_{s\theta}$ are the middle surface strains, b_s and b_θ the rotations and k_{ss}, $k_{\theta\theta}$ and $k_{s\theta}$ the curvature changes. The n's, m's, and q's are the direct (membrane) stresses, bending moments, and shears, respectively. Also r_s and r_θ are the principal radii of curvature, ν is Poisson's ratio and

$$\Omega = \omega R(\rho/E)^{1/2}$$
$$\epsilon^2 = h^2/[12R^2(1 - \nu^2)] << 1 \tag{10}$$
$$f(\sigma) = r_\theta{}^{-1} \cot \phi, \qquad M(\sigma) = mr_\theta{}^{-1} \csc \phi$$

where ω is the frequency, ρ the mass density, E Young's modulus, h shell thickness (assumed constant), R a length characteristic of the radii of curvature, m the circumferential wave number and ϕ the angle between the normal to the shell and the axial direction. Primes denote differentiation with respect to σ, which is dimensionless arc length along a meridian.

The boundary conditions at an edge have been given by Sanders [5] and consist of prescribing

$$\begin{matrix} n_{ss} & \text{or} & u \\ N_{s\theta} & \text{or} & v \\ Q_s & \text{or} & w \\ m_{ss} & \text{or} & D \end{matrix} \tag{11}$$

where

$$N_{s\theta} = n_{s\theta} + \varepsilon^2 1/2(1 - \nu^2)(3r_\theta^{-1} - r_s^{-1})\, m_{s\theta}$$

$$Q_s = q_s + Mm_{s\theta} \qquad (12)$$

$$D = w' = -b_s + ur_s^{-1}$$

The principle of conservation of energy for the vibrating shell states that

$$E_K - E_S - E_B = 0$$

where

$$E_K = \Omega^2 \int (u^2 + v^2 + w^2)\, r_\theta \sin\phi\, d\sigma = \Omega^2 K$$

$$E_S = (1 - \nu^2)^{-1} \int (n_{ss}\gamma_{ss} + n_{\theta\theta}\gamma_{\theta\theta} + 2n_{s\theta}\gamma_{s\theta})\, r_\theta \sin\phi\, d\sigma \qquad (13)$$

$$E_B = \varepsilon^2 \int (m_{ss}k_{ss} + m_{\theta\theta}k_{\theta\theta} + 2m_{s\theta}k_{s\theta})\, r_\theta \sin\phi\, d\sigma$$

and the integrals are extended over a meridian.

We shall now describe the approximations that we shall use for the solutions of this system. The system is linear, of eighth order and has singularities where $\sin\phi = 0$.[2] We limit ourselves to the case where

$$m^2\varepsilon \ll 1.$$

Four of the eight solutions vary rapidly with σ (i.e. along a meridian) and are called bending solutions, and four vary much more slowly and are called membrane solutions. The approximations for the bending solutions are quite different from the approximations for the membrane solutions. Two solutions of each type are singular where $\sin\phi = 0$.

We shall now list the approximations to the four bending solutions, first near $\sin\phi = 0$, then for $\sin\phi \neq 0$. The latter are

[2] We assume that $\sin\phi = 0$ at, and only at, the axis, and that the apex of the dome is of second degree.

linear combinations of the approximations obtained in [4] for the case $\Omega r_\theta < 1$.

For $\sin\phi = 0$:

$$w = A_1 \mathrm{ber}_m(x) + A_2 \mathrm{bei}_m(x) + A_3 \mathrm{ker}_m(x) + A_4 \mathrm{kei}_m(x) \tag{14}$$

$$x = (1 - \Omega^2)^{1/4}\, \lambda\phi \;, \qquad \lambda = \varepsilon^{-1/2} \gg 1 \tag{15}$$

For $\sin\phi \neq 0$:

$$\begin{aligned}
\begin{bmatrix} w \\ n_{\theta\theta} \\ b_\theta \end{bmatrix} &= H \begin{bmatrix} 1 \\ G_n \\ M \end{bmatrix} \{A_1 e^{\gamma} c(a) + A_2 e^{\gamma} s(a) + A_3 \pi e^{-\gamma} c(a^*) - A_4 \pi e^{-\gamma} s(a^*)\} \\
\begin{bmatrix} u \\ n_{ss} \\ n_{s\theta} \\ q_s \end{bmatrix} &= \Lambda^{-1} H \begin{bmatrix} -G_u \\ fG_n \\ MG_n \\ \Lambda^4 \end{bmatrix} \{A_1 e^{\gamma} s(a^*) - A_2 e^{\gamma} c(a^*) + A_3 \pi e^{-\gamma} s(a) + A_4 \pi e^{-\gamma} c(a)\} \\
\begin{bmatrix} b_s \\ m_{s\theta} \end{bmatrix} &= \Lambda H \begin{bmatrix} -1 \\ (1 - \nu) M \end{bmatrix} \{A_1 e^{\gamma} c(a^*) + A_2 e^{\gamma} s(a^*) - A_3 \pi e^{-\gamma} c(a) + \\
&\qquad + A_4 \pi e^{-\gamma} s(a)\} \\
\begin{bmatrix} v \\ m_{ss} \\ m_{s\theta} \\ q_\theta \end{bmatrix} &= \Lambda^2 H \begin{bmatrix} -\Lambda^{-4} M G_v \\ -1 \\ -\nu \\ M \end{bmatrix} \{-A_1 e^{\gamma} s(a) + A_2 e^{\gamma} c(a) + A_3 \pi e^{-\gamma} s(a^*) + \\
&\qquad + A_4 \pi e^{-\gamma} c(a^*)\}
\end{aligned} \tag{16}$$

where the A's are arbitrary constants and

$$x \equiv \lambda \int_{\sigma'=0}^{\sigma'=\sigma} (r_\theta^{-2} - \Omega^2)^{1/4}\, d\sigma' \tag{17}$$

$$\Lambda \equiv dx/d\sigma = \lambda (r_\theta^{-2} - \Omega^2)^{1/4} \gg 1 \tag{18}$$

$$H \equiv [(1 - \Omega^2)\varepsilon]^{1/4} \;(2\pi r_\theta \sin\phi)^{-1/2} (r_\theta^{-2} - \Omega^2)^{-3/8}$$

$$\gamma \equiv 2^{-1/2} x \;, \qquad a \equiv \gamma - (\pi/8) + (1/2) m\pi \;, \quad a^* \equiv a + (1/4)\pi$$

$c(a) \equiv \cos a, \quad s(a) \equiv \sin a$

$c(a^*) \equiv \cos a^* \quad s(a^*) \equiv \sin a^*$

$G_u \equiv r_s^{-1} + \nu r_\theta^{-1}, \quad G_n \equiv (1 - \nu^2) r_\theta^{-1}$

$G_v \equiv (2 + \nu) r_\theta^{-1} - r_s^{-1}$

In deriving these formulas we have assumed that σ is measured from the apex of the dome. Also, R is chosen as the common value of the principal radii of curvature at the apex. Then $r_s(0) = r_\theta(0) = 1$ and the definition of x for $\sin\phi = 0$ is a continuation of that for $\sin\phi \neq 0$.

These approximate formulas were obtained by an asymptotic analysis of the fundamental system of equations. The analysis indicates, and we shall assume henceforth, that the errors in these approximations are all $O(\lambda^{-1})$. For example, we could write

$$w = A_1 He^\gamma c(a) [1 + \lambda^{-1} n_w^{(1)}] + A_2 He^\gamma s(a) [1 + \lambda^{-1} n_w^{(2)}] + \ldots$$

$$n_w^{(j)} = O(1); \quad j = 1, 2, 3, \ldots$$

and similarly for all the other variables. General expressions for the corrections are not known, nor do we know even the leading terms in their expansions in λ^{-1} although these could presumably be found. For the sake of brevity we shall refrain from indicating these correction terms in our equations, upon the understanding that they are generally $O(\lambda^{-1})$. However, there are several, rather important, equations in which the leading terms cancel, and in those we shall write down symbols for the correction terms even though we don't know them.

In general we assume that terms which are $O(\lambda^{-1})$ are negligible compared with those that are $O(1)$. This gives us a rationale for deciding when these bending solutions are approximately static, for, expanding

all the approximations in powers of Ω^2, we see that the static asymptotic approximations are obtained when

$$\Omega^2 \leq 0\ (\lambda^{-2}). \tag{19}$$

The approximations we shall use for the membrane solutions are those described in [4]. At this point we assume that

$$\Omega^2 \leq 0\ (\lambda^{-1}). \tag{20}$$

Then the membrane solutions separate into two groups. Two (labelled 5 and 6) are approximately the static membrane solutions. Two (labelled 7 and 8) are (for $m \geq 2$) approximately the inextensional solutions. When (20) is satisfied, all the quantities associated with these four solutions are approximately static (i.e. independent of Ω) and 0(1) except the direct stresses of the inextensional solutions, which are of the form

$$n = A_7[\Omega^2 n_7^{(\Omega)} + \varepsilon^2 n_7^{(\varepsilon)}] + A_8[\Omega^2 n_8^{(\Omega)} + \varepsilon^2 n_8^{(\varepsilon)}]. \tag{21}$$

where $n_{7,8}^{(\Omega)}$, $n_{7,8}^{(\varepsilon)}$ are 0(1), independent of Ω, and are found by solving the motion equations as a non-homogeneous set of three equations in which the quantities multiplying Ω^2 and ε^2 are known in advance from the conditions of zero strain.

Of the eight solutions, four are singular at $\phi = 0$ and must be discarded for a dome. We may take two of these as the membrane solution numbered 6 and the inextensional solution numbered 8, and we see from (14) that the remaining two are the bending solutions numbered 3 and 4. Thus we must take

$$A_3 = A_4 = A_6 = A_8 = 0.$$

The approximate general solution can now be written

$$
\begin{aligned}
w &= He^{\gamma}\{A_1c(a) + A_2s(a)\} + A_5w^{(5)} + A_7w^{(7)} \\
u &= \Lambda^{-1}G_uHe^{\gamma}\,\{-A_1s(a^*) + A_2c(a^*)\} + A_5u^{(5)} + A_7u^{(7)} \\
v &= \Lambda^{-2}MG_vHe^{\gamma}\,\{A_1s(a) - A_2c(a)\} + A_5v^{(5)} + A_7v^{(7)} \\
b_s &= \Lambda He^{\gamma}\,\{-A_1c(a^*) - A_2s(a^*)\} + A_5b_s{}^{(5)} + A_7b_s{}^{(7)} \\
b_\theta &= MHe^{\gamma}\,\{A_1c(a) + A_2s(a)\} + A_5b_\theta{}^{(5)} + A_7b_\theta{}^{(7)} \\
m_{ss} &= \Lambda^2He^{\gamma}\,\{A_1s(a) - A_2c(a)\} + A_5m_{ss}{}^{(5)} + A_7m_{ss}{}^{(7)} \\
m_{\theta\theta} &= \Lambda^2\nu He^{\gamma}\,\{A_1s(a) - A_2c(a)\} + A_5m_{\theta\theta}{}^{(5)} + A_7m_{\theta\theta}{}^{(7)} \\
m_{s\theta} &= \Lambda M(1-\nu)\,He^{\gamma}\,\{A_1c(a^*) + A_2s(a^*)\} + A_5m_{s\theta}{}^{(5)} + A_7m_{s\theta}{}^{(7)} \\
q_s &= \Lambda^3He^{\gamma}\,\{A_1s(a^*) - A_2c(a^*)\} + A_5q_s{}^{(5)} + A_7q_s{}^{(7)} \\
q_\theta &= \Lambda^2MHe^{\gamma}\,\{-A_1s(a) + A_2c(a)\} + A_5q_\theta{}^{(5)} + A_7q_\theta{}^{(7)} \\
n_{ss} &= \Lambda^{-1}fG_nHe^{\gamma}\,\{A_1s(a^*) - A_2c(a^*)\} + A_5n_{ss}{}^{(5)} + A_7\{\Omega^2n_{ss}{}^{(\Omega)} + \lambda^{-4}n_{ss}{}^{(\epsilon')}\} \\
n_{\theta\theta} &= G_nHe^{\gamma}\,\{A_1c(a) + A_2s(a)\} + A_5n_{\theta\theta}{}^{(5)} + A_7\{\Omega^2n_{\theta\theta}{}^{(\Omega)} + \lambda^{-4}n_{\theta\theta}{}^{(\epsilon')}\} \\
n_{s\theta} &= \Lambda^{-1}MG_nHe^{\gamma}\,\{A_1s(a^*) - A_2c(a^*)\} + A_5n_{s\theta}{}^{(5)} + A_7\{\Omega^2n_{s\theta}{}^{(\Omega)} + \lambda^{-4}n_{s\theta}{}^{(\epsilon')}\}
\end{aligned}
\qquad (22)
$$

These approximations are accurate when $\Omega^2 \leq O(\lambda^{-1})$, $n \geq 2$ and $\sin\phi \neq 0$. When $\Omega^2 \leq O(\lambda^{-2})$, all the quantities are approximately static (independent of Ω) except the direct stresses of the inextensional solution. These formulas form the basis for our analysis of low frequencies and modes.

3. CALCULATION OF LOW FREQUENCIES AND MODES

In this Section we shall put the general solution (22) into various sets of edge conditions and calculate the natural modes and frequencies. The derivation will be carried out in detail for two cases but results will be given for the rest. We shall begin with the case of a completely free edge and then consider succesively "tighter" sets of edge conditions until the frequency is increased above the range, $\Omega^2 \leq O(\lambda^{-1})$ in which (22) is applicable.

We assume the edge is at $\sigma = \sigma_0$, and $\sin \phi(\sigma_0) \neq 0$. A new set of constants, B_j, $j=1, 2, 5, 7$ is introduced, defined by

$$B_1 = A_1 H(\sigma_0) e^{\gamma(\sigma_0)}, \qquad B_2 = A_2 H(\sigma_0) e^{\gamma(\sigma_0)}$$

$$B_5 = A_5, \qquad B_7 = A_7,$$

and we also set

$$\lambda^{-4} n_{ss}^{(\varepsilon)} = \Lambda^{-4} r_\theta^{-2} n_{ss}^{(\varepsilon)} \equiv \Lambda^{-4} n_{ss}^{(\lambda)}$$
$$\lambda^{-4} n_{s\theta}^{(\varepsilon)} = \Lambda^{-4} r_\theta^{-2} n_{s\theta}^{(\varepsilon)} \equiv \Lambda^{-4} n_{s\theta}^{(\lambda)}$$

In deducing the natural frequency and evaluating the constants it is to be understood that all quantities are evaluated at $\sigma = \sigma_0$.

Case (I): Free Edge. $n_{ss} = N_{s\theta} = Q_s = m_{ss} = 0$ at $\sigma = \sigma_0$

The four conditions are (keeping the leading terms only)

$$\begin{aligned} n_{ss} &= B_1 \Lambda^{-1} f G_n s(a^*) - B_2 \Lambda^{-1} f G_n c(a^*) + B_5 n_{ss}^{(5)} \\ &\quad + B_7 \{\Omega^2 n_{ss}^{(\Omega)} + \Lambda^{-4} n_{ss}^{(\lambda)}\} = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} N_{s\theta} &= B_1 \Lambda^{-1} M G_n s(a^*) - B_2 \Lambda^{-1} M G_n c(a^*) + B_5 n_{s\theta}^{(5)} \\ &\quad + B_7 \{\Omega^2 n_{s\theta}^{(\Omega)} + \Lambda^{-4} [n_{s\theta}^{(\lambda)} + g m_{s\theta}^{(7)}]\} = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} Q_s &= B_1 \Lambda^3 s(a^*) - B_2 \Lambda^3 c(a^*) + B_5 [q_s^{(5)} + M m_{s\theta}^{(5)}] \\ &\quad + B_7 \{q_s^{(7)} + M m_{s\theta}^{(7)}\} = 0 \end{aligned} \tag{25}$$

$$m_{ss} = B_1\Lambda^2 s(a) - B_2\Lambda^2 c(a) + B_5 m_{ss}^{(5)} + B_7 m_{ss}^{(7)} = 0 \tag{26}$$

where

$$g = 1/2 r_\theta^{-2}(1 - \nu^2)(3r_\theta^{-1} - r_s^{-1})$$

If we eliminate B_1 using the condition (23), we obtain a three-square system that can be written in matrix form,

$$\begin{bmatrix} n_{11} & X_{12} & (\Omega^2 X_{13} + \Lambda^{-4} X_{14}) \\ n_{21} & X_{22} & (\Omega^2 X_{23} + \Lambda^{-4} X_{24}) \\ 2^{-1/2} & X_{32}\Lambda^{-1} & (\Omega^2 \Lambda^{-1} X_{33} + \Lambda^{-4} X_{34}) \end{bmatrix} \begin{bmatrix} B_2 \Lambda^{-2} \\ B_5 \\ B_7 \end{bmatrix} = [0]$$

where

$$X_{12} = n_{s\theta}^{(5)} - n_{ss}^{(5)}(M/f), \qquad X_{22} = -n_{ss}^{(5)}/(fG_n), \qquad X_{32} = -X_{22}s(a)$$

$$X_{13} = n_{s\theta}^{(\Omega)} - n_{ss}^{(\Omega)}(M/f), \qquad X_{23} = -n_{ss}^{(\Omega)}/(fG_n), \qquad X_{33} = -X_{23}s(a)$$

$$X_{14} = n_{s\theta}^{(\lambda)} + g m_{s\theta}^{(7)} - n_{ss}^{(7)}(M/f)$$

$$X_{24} = -\{n_{ss}^{(\lambda)}/fG_n\} + q_s^{(7)} + M m_{s\theta}^{(7)}$$

$$X_{34} = -m_{ss}^{(7)} s(a^*).$$

When B_1 is eliminated, the leading terms in the coefficients of B_2 cancel in equations (24) and (25). The dominant terms in these coefficients then arise from later terms in the asymptotic expansions of n_{ss}, $n_{s\theta}$ and q_s for the bending solutions. These are not known explicitly, but we know their orders of magnitude and designate the unknown functions n_{11} and n_{21}, both of which are $O(1)$.

The frequency is found by annulling the determinant of this system, with the result

$$\Omega^2\Lambda^4 = \frac{X_{14}X_{22} - X_{12}X_{24} + 2^{1/2}(n_{21}X_{12} - n_{11}X_{22})X_{34}}{X_{12}X_{23} - X_{13}X_{22}} = \alpha_I^{\,2} \tag{27}$$

The ratios of the coefficients are found to be

$$\begin{aligned} B_1/B_7 &= \Lambda^{-2}2^{1/2}m_{ss}^{(7)}c(a^*) \\ B_2/B_7 &= \Lambda^{-2}2^{1/2}m_{ss}^{(7)}s(a^*) \\ B_5/B_7 &= \Lambda^{-4}\beta_5 \end{aligned} \tag{28}$$

where $\beta_5 = O(1)$ is a constant that may be determined from the system.

The denominator in the frequency condition (27) is

$$X_{12}X_{23} - X_{13}X_{22} = n_{ss}^{(5)}n_{s\theta}^{(\Omega)} - n_{s\theta}^{(5)}n_{ss}^{(\Omega)}$$

and cannot vanish because the direct stresses associated with the membrane and inextensional solutions must be linearly independent. Thus in the range $\Omega^2 \leq O(\lambda^{-1})$ the frequency condition can be satisfied only when

$$\Omega = \alpha_I \Lambda^{-2}$$

Since $\Lambda = \lambda r_\theta^{-1/2}$ whenever $\Omega^2 \leq O(\lambda^{-1})$, we find that there is only one natural frequency for each $m \geq 2$ in the range $\Omega^2 \leq O(\varepsilon^{1/2})$, and it is given by

$$\Omega = \alpha_I \varepsilon r_\theta(\sigma_0). \tag{29}$$

We cannot determine α_I and β_5 because we do not know n_{11} and n_{21}, which are found from the second terms in the asymptotic expansions of n_{ss}, $n_{s\theta}$ and q_s. Hence (29) is not of much practical value in calculating the frequency.

However, a first approximation to the mode is completely determined by the coefficients obtained in (28), even though we do not know β_5 precisely.

$$\left.\begin{aligned} &w = w^{(7)}(\sigma), \quad v = v^{(7)}(\sigma), \quad u = u^{(7)}(\sigma) \\ &b_s = b_s^{(7)}(\sigma), \qquad b_\theta = b_\theta^{(7)}(\sigma) \\ &\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} = X \begin{bmatrix} 1 \\ \nu \end{bmatrix} \sin\{\zeta - (\pi/4)\} + \begin{bmatrix} m_{ss}^{(7)}(\sigma) \\ m_{\theta\theta}^{(7)}(\sigma) \end{bmatrix} \end{aligned}\right\}$$

$$
\left.\begin{aligned}
m_{s\theta} &= m_{s\theta}^{(7)}(\sigma) \\
q_s &= \Lambda(\sigma)\,\chi \sin\zeta \\
q_\theta &= -\chi M(\sigma) \sin\{\zeta - (\pi/4)\} + q_\theta^{(7)}(\sigma) \\
\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} &= \Lambda^{-3}(\sigma)\chi\, G_n(\sigma) \begin{bmatrix} f(\sigma) \\ M(\sigma) \end{bmatrix} \sin\zeta \\
n_{\theta\theta} &= \Lambda^{-2}(\sigma)\,\chi G_n(\sigma) \cos\{\zeta - (\pi/4)\} \\
\chi &= 2^{1/2}\frac{H(\sigma)}{H(\sigma_0)} \left\{\frac{\Lambda(\sigma)}{\Lambda(\sigma_0)}\right\}^2 e^{\zeta} m_{ss}^{(7)}(\sigma_0) = \\
&= 2^{1/2} \left\{\frac{r_\theta(\sigma_0)}{r_\theta(\sigma)}\right\}^{3/4} \left\{\frac{\sin\phi(\sigma_0)}{\sin\phi(\sigma)}\right\}^{1/2} e^{\zeta} m_{ss}^{(7)}(\sigma_0)
\end{aligned}\right\} \quad (30)
$$

$$
\zeta = 2^{-1/2}(x - x_0) = -2^{-1/2}\lambda \int_\sigma^{\sigma_0} \{r_\theta(\sigma')\}^{-1/2}\, d\sigma' \qquad (31)
$$

It is noteworthy that all the quantities occurring in these formulas for the mode are static and relatively easy to evaluate. The displacements and rotations are dominated by the inextensional solution, the membrane solution is entirely negligible, and the bending (edge-effect) solutions have a strong influence on the stress-like quantities, making possible the satisfaction of all boundary conditions.

Although this procedure has yielded only an order-of-magnitude estimate for the frequency, it has delivered an estimate of the mode that is both more general and more complete than any previously known.

Case (II): $n_{ss} = N_s = Q_s = D = 0$ at $\sigma = \sigma_0$

Among the possible edge conditions this is the "freest" except for the free edge of Case (I). The analysis strongly resembles that of Case (I), and we shall merely record the results. Only one frequency is found in

the range $\omega \le O(\lambda^{-1})$ for each $m \ge 2$, namely

$$\Omega = \alpha_{II}\Lambda^{-3/2} = \alpha_{II}\{\varepsilon r_\theta(\sigma_0)\}^{3/4}$$

$\alpha_{II} = O(1)$ cannot be found explicitly because of cancellation of the leading terms, as was true of α_I. A complete first-approximation for the mode is found,

$$\left.\begin{aligned}
&w = w^{(7)}(\sigma), \qquad u = u^{(7)}(\sigma) \qquad v = v^{(7)}(\sigma)\\
&b_\theta = b_\theta^{(7)}(\sigma)\\
&b_s = \chi\cos\zeta + b_s^{(7)}(\sigma)\\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} = -\Lambda(\sigma)\chi \begin{bmatrix} 1 \\ \nu \end{bmatrix} \sin\{\zeta - (\pi/4)\}\\
&m_{s\theta} = -(1-\nu)M(\sigma)\,\chi\cos\zeta + m_{s\theta}^{(7)}(\sigma)\\
&q_s = -\Lambda^2(\sigma)\chi\,\sin\zeta\\
&q_\theta = \Lambda(\sigma)M(\sigma)\chi\,\sin\{\zeta - (\pi/4)\}\\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} = -\Lambda^{-2}(\sigma)G_n(\sigma)\chi \begin{bmatrix} f \\ M \end{bmatrix} \sin\zeta\\
&n_{\theta\theta} = -\Lambda^{-1}(\sigma)G_n(\sigma)\chi\,\cos\{\zeta - (\pi/4)\}
\end{aligned}\right\} \qquad (32)$$

where ζ is defined as in (31), and

$$\chi = \frac{H(\sigma)}{H(\sigma_0)}\,\frac{\Lambda(\sigma)}{\Lambda(\sigma_0)}\,D^{(7)}(\sigma_0)e^{\zeta} = \frac{r_\theta^{1/4}(\sigma_0)}{r_\theta^{1/4}(\sigma)}\,\frac{\sin^{1/2}\phi(\sigma_0)}{\sin^{1/2}\phi(\sigma)}\,D^{(7)}(\sigma_0)e^{\zeta}$$

The frequency is somewhat higher than in Case (I). The modal displacements are wholly inextensional, and the stress-like quantities are almost entirely derived from the bending solutions.

We see that Cases (I) and (II) are quite similar. In neither case can we calculate the frequency directly, but in both cases we have very good knowledge of the mode. However, if we use Rayleigh's Principle, we can

translate accurate information about the mode into accurate information about the frequency. This is exactly what Rayleigh did for spherical domes and cylinders, and we shall derive general formulas for domes with the edge conditions of these two cases in Section 4.

Case (III): $n_{ss} = N_s = w = m_{ss} = 0$ at $\sigma = \sigma_0$

This is the "freest" of the remaining boundary conditions. The analysis proceeds as in Case (I) except that (25) is replaced by

$$w = B_1 c(a) + B_2 s(a) + B_5 w^{(5)} + B_7 w^{(7)} = 0.$$

After eliminating B_1, the matrix equation of the system is

$$\begin{bmatrix} n_{11} & X_{12} & (\Omega^2 X_{13} + \Lambda^{-4} X_{14}) \\ 2^{-1/2}\Lambda^2 & \Lambda X_{22} & (\Omega^2 \Lambda X_{23} + X_{24}) \\ 2^{-1/2} & \Lambda^{-1} X_{32} & (\Omega^2 \Lambda^{-1} X_{33} + \Lambda^{-4} X_{34}) \end{bmatrix} \begin{bmatrix} \Lambda^{-2} B_2 \\ B_5 \\ B_7 \end{bmatrix} = [0]$$

where

$$X_{22} = -\frac{n_{ss}^{(5)} c(a)}{fG_n}, \qquad X_{23} = -\frac{n_{ss}^{(\Omega)} c(a)}{fG_n}, \qquad X_{24} = w^{(7)} s(a*)$$

$$X_{32} = \frac{n_{ss}^{(5)} s(a)}{fG_n}, \qquad X_{33} = \frac{n_{ss}^{(\Omega)} s(a)}{fG_n}$$

and the remaining X's are defined as in Case (I). We find for the frequency

$$\Omega^2 \Lambda = \frac{X_{12} X_{24}}{X_{12}(X_{33} - X_{23}) + X_{13}(X_{22} - X_{32})} = \alpha_{III}^2$$

and after some reduction

$$\alpha_{III}^2 = 2^{-1/2} \left[w^{(7)} G_n \left\{ \frac{f n_{s\theta}^{(5)} - n_{ss}^{(5)}}{n_{ss}^{(\Omega)} n_{s\theta}^{(5)} - n_{s\theta}^{(\Omega)} n_{ss}^{(5)}} \right\} \right]_{\sigma = \sigma_0} \qquad (33)$$

$$\Omega = \alpha_{III}\Lambda^{-1/2} = \alpha_{III}\{\varepsilon r_\theta(\sigma_0)\}^{1/4}$$

The frequency is now higher by a factor of roughly $\varepsilon^{-1/2}$ than in Case (II). The mode is

$$\left.\begin{aligned}
&w = w^{(7)}(\sigma) - \chi\cos\zeta\,, \qquad u = u^{(7)}(\sigma)\,, \qquad v = v^{(7)}(\sigma)\\
&b_s = \Lambda(\sigma)\chi\cos\{\zeta + (\pi/4)\}\,, \qquad b_\theta = b_\theta^{(7)}(\sigma) - M\chi\cos\zeta\\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} = -\Lambda^2(\sigma)\chi\begin{bmatrix} 1 \\ \nu \end{bmatrix}\sin\zeta\,, \qquad m_{s\theta} = -\Lambda(\sigma)M(1-\nu)\chi\cos\{\zeta + (\pi/4)\}\\
&q_s = -\Lambda^3(\sigma)\chi\sin\{\zeta + (\pi/4)\}\,, \qquad q_\theta = \Lambda^2(\sigma)M\chi\sin\zeta\\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} = -\Lambda(\sigma)G_n(\sigma)\chi\begin{bmatrix} f \\ M \end{bmatrix}\sin\{\zeta + (\pi/4)\}\\
&\qquad + \Lambda^{-1}(\sigma_0)w^{(7)}(\sigma_0)G_n(\sigma_0)2^{-1/2}\left\{L_S\begin{bmatrix} n_{ss}^{(S)}(\sigma) \\ n_{s\theta}^{(S)}(\sigma) \end{bmatrix} + L_\Omega\begin{bmatrix} n_{ss}^{(\Omega)}(\sigma) \\ n_{s\theta}^{(\Omega)}(\sigma) \end{bmatrix}\right\}\\
&n_{\theta\theta} = -G_n\chi\cos\zeta
\end{aligned}\right\} \tag{34}$$

where

$$\chi = \{H(\sigma)/H(\sigma_0)\}e^{\zeta}w^{(7)}(\sigma_0)$$

$$L_S = \left[\frac{Mn_{ss}^{(\Omega)} - fn_{s\theta}^{(\Omega)}}{n_{ss}^{(\Omega)}n_{s\theta}^{(S)} - n_{s\theta}^{(\Omega)}n_{ss}^{(S)}}\right]_{\sigma=\sigma_0}$$

$$L_\Omega = \left[\frac{fn_{s\theta}^{(S)} - Mn_{ss}^{(S)}}{n_{ss}^{(\Omega)}n_{s\theta}^{(S)} - n_{s\theta}^{(\Omega)}n_{ss}^{(S)}}\right]_{\sigma=\sigma_0}$$

This mode differs from those in the two preceding cases in two important ways. First, the displacements are no longer completely inextensional, for the bending solutions make a contribution to w near the edge. Second the effect of the membrane solution is not now completely negligible but is felt in the formulas for the direct stresses.

Case (IV): $m_{ss} = N_{s\theta} = w = D = 0$ at $\sigma = \sigma_0$

The analysis is the same in this case as in Case (III) with an obvious change in the last boundary condition. The natural frequency is found to be

$$
\begin{aligned}
&\Omega^2 = \alpha_{IV}{}^2 \Lambda^{-1} \\
&\alpha_{IV}{}^2 = 2^{1/2} \left[w^{(7)} G_n \left\{ \frac{f n_{s\theta}{}^{(5)} - M n_{ss}{}^{(5)}}{n_{ss}{}^{(\Omega)} n_{s\theta}{}^{(5)} - n_{s\theta}{}^{(\Omega)} n_{ss}{}^{(5)}} \right\} \right]_{\sigma = \sigma_0}
\end{aligned}
\tag{35}
$$

and the mode is

$$
\left.
\begin{aligned}
&w = w^{(7)}(\sigma) + \chi \sin\{\zeta - (\pi/4)\} \ , \qquad u = u^{(7)}(\sigma) \ , \qquad v = v^{(7)}(\sigma) \\
&b_s = -\Lambda(\sigma)\chi \sin \zeta \ , \qquad b_\theta = b_\theta{}^{(7)}(\sigma) + \chi \sin\{\zeta - (\pi/4)\} \\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} = -\Lambda(\sigma)\chi \begin{bmatrix} 1 \\ \nu \end{bmatrix} \cos\{\zeta - (\pi/4)\} \ , \qquad m_{s\theta} = \Lambda(\sigma) M \chi (1-\nu) \sin\zeta \\
&q_s = -\Lambda^2(\sigma)\chi \cos \zeta \ , \qquad q_\theta = \Lambda^2(\sigma) M \chi \cos\{\zeta - (\pi/4)\} \\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} = -\Lambda^{-1}(\sigma) G_n(\sigma) \begin{bmatrix} f \\ M \end{bmatrix} \chi \cos \zeta \\
&\qquad + \Lambda^{-1}(\sigma_0) w^{(7)}(\sigma_0) G_n(\sigma_0) 2^{-1/2} \left\{ L_5 \begin{bmatrix} n_{ss}{}^{(5)}(\sigma) \\ n_{s\theta}{}^{(5)}(\sigma) \end{bmatrix} \right. \\
&\qquad \left. + L_\Omega \begin{bmatrix} n_{ss}{}^{(\Omega)} \\ n_{s\theta}{}^{(\Omega)} \end{bmatrix} \right\} \\
&n_{\theta\theta} = G_n(\sigma)\chi \sin\{\zeta - (\pi/4)\}
\end{aligned}
\right\}
\tag{36}
$$

where

$$
\chi = 2^{1/2}\{H(\sigma)/H(\sigma_0)\}\, e^{\zeta} w^{(7)}(\sigma_0)
$$

This frequency and mode are qualitatively much like those in Case (III). We see from (35) and (33) that the frequency estimate in the present case is larger than in Case (III) by a simple factor $2^{1/2}$.

The edge conditions considered in Cases (I) - (IV) all have $n_{ss} = N_{s\theta} = 0$ at $\sigma = \sigma_0$, ie the edges of the shell have been free to move in directions tangent to the middle surface. We have now exhausted all the cases with this property. If we work out similar analyses for

Case (V): $n_{ss} = v = Q_s = m_{ss} = 0$ at $\sigma = \sigma_0$

Case (VI): $u = N_{s\theta} = Q_s = m_{ss} = 0$ at $\sigma=\sigma_0$,

we see that natural frequencies in the range $\Omega^2 \geq O(\lambda^{-1})$ cannot occur, i.e. for these edge conditions all the natural frequencies obey

$$\Omega^2 \geq O(1).$$

But all the remaining edge conditions are obtained from (V) or (VI) by "tightening" some of the conditions. Hence in all the remaining cases the natural frequencies are at least as high as in Cases (V) or (VI). We conclude that only for Cases (I) - (IV) can we find natural frequencies in the range

$$\Omega^2 \leq O(\varepsilon^{1/2}).$$

Now it is easy to see from the motion equations that inextensional solutions, i.e. solutions having the property that

$$n_{ss},\ n_{\theta\theta},\ n_{s\theta} \leq O(\lambda^{-1})$$

and all other quantities are $O(1)$, cannot occur when $\Omega^2 \geq O(1)$. Hence for a dome inextensional modes and frequencies can occur only in Cases (I) - (IV), i.e. only when the edge is free to move tangentially.

4 APPLICATIONS OF RAYLEIGH'S PRINCIPLE

In this Section we shall see how estimates of the inextensional frequencies can be obtained for the Cases (I) and (II) by using Raleigh's Principle.

Rayleigh's Principle states that for any field of displacements satisfying the edge conditions on displacements,

$$\Omega^2 \leq \Omega_E{}^2 = (E_S + E_B) / K \tag{37}$$

where E_S, E_B and K are to be calculated from the given field of displacements by means of (13). The accuracy of the estimated frequency, Ω_E, depends on (and is usually much better than) the accuracy of the assumed displacement field. We must emphasize (because it is occasionally overlooked) the effect of the edge conditions on the displacements. If these edge conditions are not satisfied by the chosen displacement field, Ω_E may differ wildly from Ω and need not even be the larger of the two.

In applying Rayleigh's Principle to Cases (I) and (II) we shall take as the trial displacements the approximate modes given for these Cases by our previous analysis. From (13), (4) and (5) we have

$$E_S = (1 - \nu^2)^{-2} \int_{\sigma=0}^{\sigma_0} \{n_{ss}{}^2 + n_{\theta\theta}{}^2 - 2\nu n_{ss} n_{\theta\theta} + 2(1 + \nu) n_{s\theta}{}^2\} \, r_\theta \sin\phi \, d\sigma \tag{38}$$

$$E_B = \lambda^{-4} (1 - \nu^2)^{-1} \int_{\sigma=0}^{\sigma_0} \{m_{ss}{}^2 + m_{\theta\theta}{}^2 - 2\nu m_{ss} m_{\theta\theta} + 2(1 + \nu) m_{s\theta}{}^2\} r_\theta \sin\phi \, d\sigma \tag{39}$$

$$K = \int_{\sigma=0}^{\sigma_0} (u^2 + v^2 + w^2) \, r_\theta \sin\phi \, d\sigma \tag{40}$$

Referring to the formulas (30) and (32) for the modes in the two cases, we see that two kinds of terms occur, namely terms of inextensional and edge-effect types. The integrals of the inextensional terms are of the same order as the terms themselves and cannot be evaluated explicitly until the shell shape is specified. The integrals of the edge effect terms are smaller by an order of magnitude than the terms themselves and can be evaluated explicitly (though approximately)

by use of the Laplace Method for asymptotic approximation of definite integrals.

For example in Case (II), (38) and (32) lead to

$$E_s = (1 - \nu^2)^{-2} \int_0^{\sigma_0} n_{\theta\theta}{}^2 r_\theta \sin\phi \, d\sigma$$

$$= (1 - \nu^2)^{-2} \int_0^{\sigma_0} [\Lambda^{-2}(\sigma)\chi^2(\sigma)\, r_\theta(\sigma)\sin\phi(\sigma) G_h{}^2(\sigma)] \cos^2 [\zeta - (\pi/4)] d\sigma$$

$$= \int_0^{\sigma_0} \{ \Lambda^{-2}(\sigma) r_\theta{}^{-1}(\sigma) \sin \phi(\sigma) \Delta(\sigma) [D^{(7)}(\sigma_0)]^2 \} \times$$

$$\times e^{2\zeta} \cos^2[\zeta - (\pi/4)]\, d\sigma$$

where

$$\Delta(\sigma) = H(\sigma)\Lambda(\sigma)/[H(\sigma_0)\Lambda(\sigma_0)]$$

The function $e^{2\zeta}$ has the value unity for $\sigma = \sigma_0$ and decreases rapidly to zero as σ decreases from σ_0. Hence this integral is of Laplace type, i.e. only the region near $\sigma = \sigma_0$ contributes appreciably to the integral. We may therefore appoximate it by

$$E_s = \{\Lambda^{-2} r_\theta{}^{-1} D^{(7)2} \sin\phi\}_{\sigma_0} (1/2) \int_0^{\sigma_0} e^{2\zeta}(1 + \sin 2\zeta)\, d\sigma$$

The integral in this expression can be evaluated approximately with the aid of (31) to give

$$E_s = \{\Lambda^{-3} r_\theta{}^{-1} D^{(7)2} \sin\phi\}_{\sigma_0} (2^{1/2}/8)$$

We know that $\Lambda^2 = O(\lambda^{-3})$, hence

$$\Lambda(\sigma_0) = \lambda r_\theta{}^{-1/2}(\sigma_0).$$

Thus, finally we find

$$E_s = (1/8)\lambda^{-3}[2r_\theta(\sigma_0)]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi(\sigma_0)$$

In a similar manner E_B may be estimated,

$$E_B = (3/8)\lambda^{-3}[2r_\theta(\sigma_0)]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi\,(\sigma_0)$$

Hence for Case (II)

$$\Omega_L{}^2 = \frac{\lambda^{-3}[r_\theta(\sigma_0)/2]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi(\sigma_0)}{K^{(7)}} \qquad (41)$$

where

$$K^{(7)} = \int_{\sigma=0}^{\sigma_0} \{[u^{(7)}(\sigma)]^2 + [v^{(7)}(\sigma)]^2 + [w^{(7)}(\sigma)]^2\} r_\theta \sin\phi d\sigma$$

Applying the same analysis in Case (I) we find

$$E_s = O(\lambda^{-5})$$

$$E_B = E_B^{(7)} + O(\lambda^{-5})$$

$$E_B^{(7)} = \lambda^{-4} \int_{\sigma=0}^{\sigma_0} \{[m_{ss}^{(7)}]^2 + [m_{\theta\theta}^{(7)}]^2 - 2\nu m_{ss}^{(7)} m_{\theta\theta}^{(7)}$$

$$+ 2(1+\nu)[m_{s\theta}^{(7)}]^2\} r_\theta \sin\phi \, d\sigma$$

$$K = K^{(7)}$$

Hence we obtain the estimate

$$\Omega^2_E = E_B^{(7)}/K^{(7)} = O(\lambda^{-4}) \qquad (42)$$

We see that in this case the estimate given by Rayleigh's Principle can be derived solely from the inextensional displacements. This is not true of the estimate just obtained in Case (II), nor is it true of the Rayleigh estimates that are obtained for the inextensional frequencies in Cases (III) and (IV). Equation (42) is of course just the estimate that Rayleigh used to find the inextensional frequencies for a spherical dome. However, neither Rayleigh nor any subsequent investigator seems to have been sure of the conditions under which the estimate is accurate. We now see that it is accurate only when the edge of the dome is free.

5. INEXTENSIONAL FREQUENCIES FOR A SPHERICAL DOME

In this Section we carry out the calculation of the two lowest inextensional frequencies for spherical dome under the edge conditions of Cases (II) and (III), using formulas (41) and (33), respectively.

For a spherical dome the inextensional solution that is finite at $\phi = 0$ has

$$u^{(7)} = v^{(7)} = \sin\phi \tan^m(\phi/2), \qquad m \geq 2$$
$$w^{(7)} = -(m + \cos\phi)\tan^m(\phi/2) \tag{43}$$
$$\beta^{(7)} = \{\sin\phi - m(m + \cos\phi)\csc\phi\}\tan^m(\phi/2)$$

and the kinetic energy is given by

$$K^{(7)} = K(\phi_0, m) = \int_0^{\phi_0} \tan^{2m}(\phi/2)\{2\sin^2\phi + (m + \cos\phi)^2\}\sin\phi \, d\phi .$$

Rayleigh has shown how this integral can be evaluated for integer values of m.

In Case (II) Equation (41) reduces to

$$\Omega_E^2 = \frac{2^{-1/2}\varepsilon^{3/2}}{dK(\phi_0,m)}\sin\phi_0 \tan^{2m}(\phi_0/2)\{\sin\phi_0 - m(m + \cos\phi_0)\csc\phi_0\}^2 \tag{44}$$

Figure 1 shows graphs of the relations between Ω_E and ϕ_0 for $m = 2$ and 3, obtained from (44).

In Case (III) the frequency is given by (33). To evaluate this for a sphere, we observe first that the membrane solution finite at $\phi = 0$ has (see Koss [5])

$$n_{s\theta}^{(5)} = -n_{ss}^{(5)},$$

and (33) reduces to

$$\Omega_{III}^2 = \frac{-2^{-1/2}(1-\nu^2)(m + \cos\phi_0)^2 \csc\phi_0 \tan^m(\phi_0/2)}{\left\{n_{ss}^{(\Omega)} + n_{s\theta}^{(\Omega)}\right\}\Big|_{\sigma_0}} \tag{45}$$

To find

$$z = n_{ss}^{(\Omega)} + n_{s\theta}^{(\Omega)}$$

we must solve the system of equations obtained from the motion equations by setting $\varepsilon = 0$ and taking for u, v and w the inextensional displacements (43). The governing equation is

$$\frac{dz}{d\phi} + (m + 2\cos\phi)\csc\phi\, z = -(1 - \nu^2)\{u + v - w(m + \cos\phi)\csc\phi\}$$

and a particular solution (which is all we need) is

$$z(\phi_0) = -(1 - \nu^2)K(\phi_0, m)\sin^{-2}\phi_0 \tan^{-m}(\phi_0/2)$$

Combining this with (45) and (33) we find

$$\Omega^2 = \frac{\varepsilon^{1/2}(m + \cos\phi)^2 \sin\phi_0 \tan^{2m}\phi_0}{2^{1/2}K(\phi_0, m)} \tag{46}$$

The frequencies predicted by (46) when m = 2 and 3 are shown in Figure 2.

The inextensional frequencies in Case (IV) are $2^{1/2}$ times those of Case (II.).

6. DISCUSSION

The results derived in the preceding Sections aid our understanding of inextensional modes and the roles they play in shell vibration problems. In brief, we may say that, when inextensional modes can occur, they are associated with frequencies lower than those associated with any other type of mode. At most one inextensional frequency is found for each $m \geq 2$, but for a dome none can occur if the edge conditions involve significant constraint against motion tangent to the shell surface. The inextensional frequencies are far more sensitive to the edge conditions than are the (higher) membrane and bending frequencies.

We have seen that the procedure used by Rayleigh to find inextensional frequencies yields very nearly the results obtained by the present method for a free edge, Case (I). For Cases (II) - (IV) the present method predicts modes with predominantly inextensional displacements, which cannot be found by Rayleigh's procedure. The frequencies in these Cases are higher than in Case (I) but still low compared to the lowest frequencies obtained for all the remaining edge conditions.

An interesting aspect of this analysis is this. When $\Omega^2 \leq O(\lambda^{-2})$, the only quantities among all the eight solutions that depend on Ω are the direct stresses associated with the inextensional solutions. When these solutions are inserted in the boundary conditions and a natural frequency is calculated, it is clear that the inextensional direct stresses are the indispensable ingredients of the calculation. For, if they are absent from the frequency equation, it does not contain the frequency and cannot be satisfied, and no natural frequencies will be found. Yet, despite the importance of these direct stresses,

they can be completely neglected in applying Rayleigh's Principle to find the inextensional frequencies for a free-edged dome.

We have not considered boundary conditions of elastic constraint at the edge. In general we may expect that these will produce frequencies lying between those associated with the two "pure" edge conditions that are combined to give the elastic condition. For example, the lowest natural frequency associated with the boundary condition

$$n_{ss} = N_{s\theta} = w = \xi m_{ss} + (1 - \xi)D = 0,$$

where $0 \leq \xi \leq 1$, should satisfy

$$\alpha_{III}\Lambda^{-1/2} \leq \Omega \leq \alpha_{IV}\Lambda^{-1/2} .$$

Although we have chosen to demonstrate this procedure for domes, it ought to work equally well for shells with two edges. However, it remains always subject to the condition that $m^2\varepsilon \ll 1$.

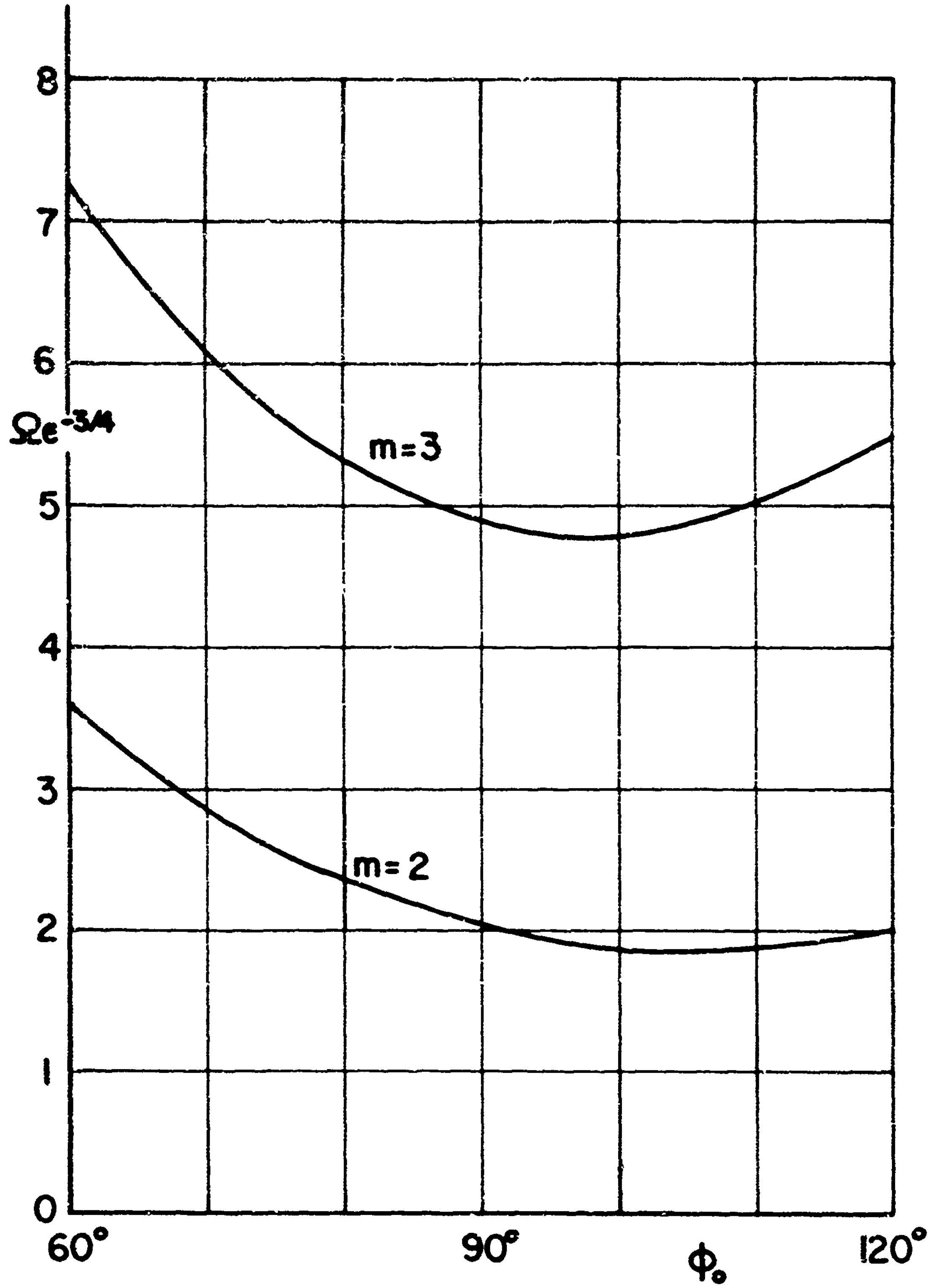

Figure 1. Inextensional Frequencies as Functions of Edge Angle, ϕ_0, for the Edge Condition $n_{ss} = N_{s\theta} = Q_s = \nu = 0$.

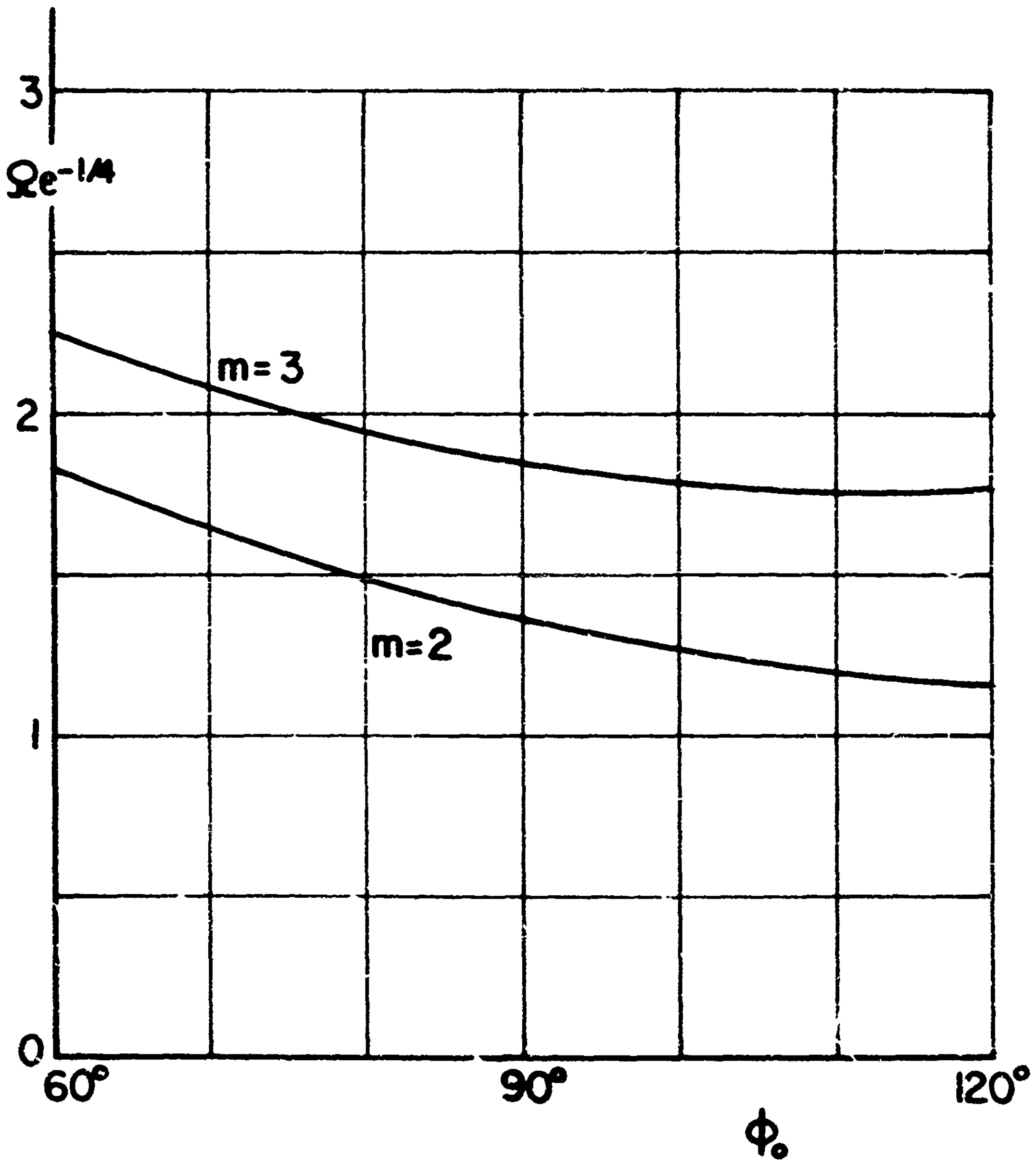

Figure 2. Inextensional Frequencies as Functions of Edge Angle, ϕ_0, for the Edge Condition $n_{ss} = N_{s\theta} = w = m_{ss} = 0$

7. REFERENCES

1. Lord Rayleigh, "The Theory of Sound", Vol. I, second edition, Dover Publications, Inc., New Yord, N.Y., 1944.

2. A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity", fourth edition, Dover Publications, Inc., New York, N.Y., 1944.

3. R.N. Arnold and G. B. Warburton, "Flexural Vibrations of the Walls of Thin Cylindrical Shells Having Freely-supported Ends", Proc. Roy. Soc. of London Series A, Vol. 197, 1949, pp. 238-256.

4. Edward W. Ross, Jr., "Approximations in Non-Symmetric Shell Vibrations", U. S. Army Materials Research Agency Report No. TR67-09, April 1967, Submitted to Journal of the Acoustical Society of America.

5. J. L. Sanders, "An Improved First-Approximation Theory for Thin Shells", NASA Report 24, 1959.

6. Edward W. Ross, Jr., "On Membrane Frequencies for Spherical Shell Vibrations" U. S. Army Natick Laboratories Report No. 67-74-OSD, May 1967. Submitted to AIAA Journal.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) U. S. Army Natick Laboratories Natick, Massachusetts	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

ON INEXTENSIONAL VIBRATIONS OF THIN SHELLS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Edward W. Ross, Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1967	28	6

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	68-14-OSD
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

This document is approved for release and sale;its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Natick Laboratories Natick, Massachusetts 01760

13. ABSTRACT

In this paper, the non-symmetric, free, elastic vibrations of thin domes of revolution are studied. It is assumed that the frequency is low. The asymptotic approximations previously given by the writer are used to estimate the general solution to the shell vibration equations at low frequencies. Approximations for the low natural frequencies and modes are derived systematically under a variety of edge conditions. Low natural frequencies are found only when the edge conditions impose no forces tangent to the shell surface. When the edge is free (and only then) Rayleigh's inextensional frequencies are recovered. For certain other edge conditions new natural frequencies are found that are above Rayleigh's frequencies but still low compared, e.g., with the lowest membrane frequency. The displacement modes associated with these new frequencies are mostly of inextensional type. The general results are applied to estimate these new frequencies for spherical domes.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified.

Security Classification

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Analysis	8					
Estimation	8					
Low frequency	9					
Vibrations	9					
Inextensional	0					
Elastic shells	9					
Spheres	9					
Asymptotic expansion	10					
Rayleigh's Principle	10					
Prediction	4					
Performance	4					
Containers	4					
Parachutes	4					
Tents	4					

Unclassified
Security Classification